1. In a geometric series the common ratio is r and sum to n terms is S_n

Given

$$S_\infty = \frac{8}{7} \times S_6$$

show that $r = \pm\frac{1}{\sqrt{k}}$, where k is an integer to be found.

(4)

$S_\infty = \frac{a}{1-r}$ $\qquad$ $S_n = \frac{a(1-r^n)}{1-r}$

$$\frac{a}{1-r} = \frac{8}{7}\left(\frac{a(1-r^6)}{1-r}\right)$$

$$\frac{a}{\cancel{1-r}} = \frac{8a(1-r^6)}{7(\cancel{1-r})}$$

$$\cancel{a} = \frac{8\cancel{a}(1-r^6)}{7}$$

$$1 = \frac{8(1-r^6)}{7}$$

$$1-r^6 = \frac{7}{8}$$

$$r^6 = \frac{1}{8}$$

$$r = \frac{\sqrt[6]{1}}{\sqrt[6]{8}}$$

$$r = \frac{1}{\sqrt{2}}$$

$$r = \pm\frac{1}{\sqrt{2}}$$

$\therefore K = 2$

2. A competitor is running a 20 kilometre race.

She runs each of the first 4 kilometres at a steady pace of 6 minutes per kilometre.
After the first 4 kilometres, she begins to slow down.

In order to estimate her finishing time, the time that she will take to complete each subsequent kilometre is modelled to be 5% greater than the time that she took to complete the previous kilometre.

Using the model,

(a) show that her time to run the first 6 kilometres is estimated to be 36 minutes 55 seconds, **(2)**

(b) show that her estimated time, in minutes, to run the rth kilometre, for $5 \leqslant r \leqslant 20$, is

$$6 \times 1.05^{r-4}$$

(1)

(c) estimate the total time, in minutes and seconds, that she will take to complete the race. **(4)**

a) Total time for first 6 km = $(6 \times 4) + (6 \times 1.05) + (6 \times 1.05^2)$ (1)

minutes   km

= 36.915 minutes = 36 minutes 55 seconds (1)

$(60 \times 0.915 = 54.9 \approx 55)$

b) 5th km : 6×1.05^1 $5-1=4$
6th km : 6×1.05^2 $6-2=4$
7th km : 6×1.05^3 $7-3=4$ (1)

∴ it follows that the time for rth km is $6 \times 1.05^{r-4}$

Geometric Series
$a = 6.3$
$r = 1.05$
$n = 16$
$S_n = \dfrac{a(1-r^n)}{1-r}$

c) total time = 24 minutes + $\sum_{r=5}^{r=20} 6 \times 1.05^{r-4}$ (1)

4km (1)

$= 24 \text{ minutes} + \dfrac{6.3(1-1.05^{16})}{1-1.05}$ (1)

= 173.042 minutes = 173 minutes 3 seconds (1)

(60×0.042)

3. A geometric series has common ratio r and first term a.

Given $r \neq 1$ and $a \neq 0$

(a) prove that

$$S_n = \frac{a(1-r^n)}{1-r}$$

(4)

$\times r$ $S_n = a + ar + ar^2 + \dots + ar^{n-1}$ $\times r$ 1

$\Rightarrow r \cdot S_n = ar + ar^2 + ar^3 + \dots + ar^n$ 1

$\Rightarrow S_n - r \cdot S_n = a + ar + ar^2 + \dots + ar^{n-1} - ar - ar^2 - \dots - ar^n$

$\Rightarrow S_n - r \cdot S_n = a - ar^n$ 1 (we now want to rearrange and manipulate this to get the required answer / proof)

$\Rightarrow S_n(1-r) = a(1-r^n)$ $\Rightarrow$ $\underline{S_n = \dfrac{a(1-r^n)}{1-r}}$ as required. 1

$\div(1-r)$

Given also that S_{10} is four times S_5

(b) find the exact value of r.

(4)

Recall that : $S_n = \dfrac{a(1-r^n)}{1-r}$

$\Rightarrow S_{10} = 4 \times S_5$

$\Rightarrow \dfrac{a(1-r^{10})}{1-r} = \dfrac{4a(1-r^5)}{1-r}$ $\div a$ $\times 1-r$ 1

$\Rightarrow 1 - r^{10} = 4(1-r^5)$ 1 $\Rightarrow 1 - r^{10} = 4 - 4r^5$

$\Rightarrow r^{10} - 4r^5 + 3 = 0$ then let $x = r^5$ and $x^2 = r^{10}$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-3)(x-1) = 0$

$\Rightarrow (r^5-3)(r^5-1) = 0$ $\rightarrow$ $r^5 = 1 \Rightarrow r = 1$ (but this solution isn't valid since $r \neq 1$).

$\Rightarrow r^5 = 3$ 1

$\Rightarrow r = \underline{\sqrt[5]{3}}$ $\Rightarrow$ The exact value of r is $\underline{r = \sqrt[5]{3}}$ 1

4. Show that

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^{n}\cos(180n)^{\circ}=\frac{9}{28}$$

(3)

$a = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$ 1

formula for sum to infinity of geometric series.

$S_\infty = \frac{a}{1-r}$

a = first term

r = common ratio

$r = -\frac{3}{4}$

$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{\frac{9}{16}}{1-\left(-\frac{3}{4}\right)}$ 1

$= \frac{\frac{9}{16}}{\frac{7}{4}}$

$= \frac{9}{28}$ $\square$ 1